MATHEMATICS

# INTRODUCTION TO THE RESIDUE CALCULUS [1])

BY

J. G. VAN DER CORPUT

(Communicated at the meeting of November 26, 1960)

The theory of complex variables contains as a special branch the residue calculus, which is based on the fact that under certain conditions a contour integral is equal to $2\pi i$ times the sum of the corresponding residues. The conditions to be imposed on the path and the integrand are rather strong. Usually we assume that the path is a closed Jordan curve, that the integrand is analytic on the path, and moreover, apart from a finite number of critical points, also analytic inside the integration path. Each of the said critical points yields a residue and this residue can be evaluated if we know the behavior of the integrand in the neighborhood of the critical point.

The characteristic feature of this calculus can be formulated as follows: For the evaluation of an integral it is, under certain circumstances, possible to introduce a finite number of critical points with the property that each of these critical points yields a certain contribution (the residue of the integrand at that point) and with the property that the integral itself is, apart from an elementary factor, equal to the sum of the residues at the critical points. For the evaluation of the residue at a critical point, it is sufficient to know the behavior of the integrand in the neighborhood of that point.

The purpose of this paper is to expose a new, much more general calculus with the same characteristic feature. In the new calculus we do not restrict ourselves to integrals along a closed contour. On the contrary, the integration domain may be an interval, a contour (closed or not), a region lying in an $n$-dimensional space, an $m$-dimensional surface lying in the $n$-dimensional space ($m < n$), and so on. Under general conditions all these integrals satisfy the principle formulated above. In this case we can choose the residues even in such a way that the elementary factor mentioned above, which in the original residue calculus has the value $2\pi i$, is in the new calculus equal to 1. The principle of the new residue calculus can therefore be formulated as follows:

*For the evaluation of an integral $j$ it is often possible to find a finite number of critical points $\alpha$, each of which yields a certain contribution*

[1]) This report has been facilitated by a grant (G 5228) of the National Science Foundation, Washington, D.C.

*(called the residue of the integrand at the critical point) which is uniquely defined by the behavior of the integration domain and of the integrand in the neighborhood of $\alpha$, with the property that the integral $j$ is equal to the sum of all these residues.*

Many formulas occurring in the theory of the special functions defined by integrals are particular cases of this general principle.

This principle is particularly important in asymptotics, that branch of mathematics where we use a complex unbounded variable $\omega$ and examine the behavior of functions of $\omega$ for large values of $|\omega|$. There we do not use the ordinary notion of equality, but that of asymptotic equality. Two functions $f(\omega)$ and $g(\omega)$ are called asymptotically equal if for each admissible value of $\omega$ with sufficiently large $|\omega|$ the functions $f(\omega)$ and $g(\omega)$ are defined in such a way that for each fixed positive $q$ the order relation

$$f(\omega)-g(\omega)=O|\omega|^{-q}$$

holds [2]).

We say that the asymptotic behavior of a function is determined if this function is asymptotically equal to a given function. In asymptotics, we are only interested in the asymptotic behavior of the functions, not in their exact values. In order to determine by means of the residue calculus the asymptotic behavior of an integral $j$ which depends on $\omega$, we introduce again a finite number of critical points $\alpha$. These points and their number may depend on $\omega$, but usually their number is fixed. For each critical point $\alpha$ we introduce a function of $\omega$, the asymptotic behavior of which is uniquely defined by the behavior of the integration domain and of the integrand in the neighborhood of $\alpha$. This function is called the asymptotic residue of the integrand at $\alpha$. Notice that not the asymptotic residue, but only its asymptotic behavior, is defined. This is sufficient, since this behavior is the only thing with which the asymptoticist is concerned. The residue principle assumes in asymptotics the following form:

*For the determination of the asymptotic behavior of an integral $j$ it is often possible to find a finite number of critical points $\alpha$, each yielding a certain contribution (called the asymptotic residue of the integrand at the critical point) the asymptotic behavior of which is uniquely defined by the behavior of the integration domain and of the integrand in the neighborhood of $\alpha$ with the property that the integral $j$ is asymptotically equal to the sum of all these residues.*

This principle yields for many simple and multiple integrals an asymptotic expansion which enables us to calculate the integral for large values of $|\omega|$ with a high degree of accuracy.

Now I must make some remarks about the theory of neutrices (Refs. [1], [2], [3], [4]), since the principle of residues is based on this theory. Often we try to write a certain number a in another form, where

---

[2]) Here "fixed" means independent of $\omega$.

the rules applied in this transformation involve a new [3]) variable $\xi$ which traverses a given set $N'$. It is possible that in this way we write $a$ as a finite sum of constant [4]) or variable [5]) terms. In view of the fact that $a$ is independent of $\xi$, we see that the sum of the variable terms obtained in this way is constant, but we have no assurance that this sum is always equal to zero. For instance, the formula

$$\log 2 = \log 2\xi - \log \xi$$

shows that we obtain a wrong result if we replace each variable term by zero. However, the neutrix calculus is based on the fact that under a general condition all the variable terms occurring in a reasoning may be neglected. To formulate this condition, we consider a certain class $N$ of real or complex functions $\nu(\xi)$ defined for each element $\xi$ of the set $N'$ mentioned above. We call all these functions $\nu(\xi)$ negligible, because it is our intention to deduce a sufficient condition that everywhere in our reasoning all these terms may be neglected completely. If we neglect first a term $\nu(\xi)$ and later a term $\nu^*(\xi)$, then in all we have neglected the sum $\nu(\xi)+\nu^*(\xi)$, so that it is natural to make the convention that the sum of two negligible functions is always negligible. Furthermore, we also make the convention that the difference of two negligible functions is always negligible. Each negligible function $\nu(\xi)$ has the property that $2\nu(\xi)=\nu(\xi)+\nu(\xi)$ is negligible, but $\nu(\xi)/2$ is not necessarily negligible.

Let us return to the constant number $a$ which we have written as a sum of constant or variable terms. Assume that all the variable terms are negligible. Let $b$ be the sum of the constant terms and let $c$ be the sum of the variable terms, so that $a=b+c$, where $b$ and $c$ are constant. Moreover, $c$ is a finite sum of negligible functions and therefore itself negligible. Since $c$ must equal zero, we introduce the following neutrix condition:

*A constant negligible function is always identically equal to zero.*

*A class $N$ of real or complex functions $\nu(\xi)$ defined for each element $\xi$ of a given set $N'$ is called a neutrix if any two functions belonging to $N$ have the property that their difference (and therefore also their sum) belongs to $N$ and if each constant function belonging to $N$ is identically equal to zero.*

$N'$ is called the domain and $\xi$ is called the variable of $N$; the functions $\nu(\xi)$ are said to be negligible (in $N$). A neutrix owes its name to the fact that it "neutralizes" the influence of the negligible functions.

Consider a neutrix $N$ with domain $N'$, with variable $\xi$ and with negligible functions $\nu(\xi)$. Consider moreover a neutrix $P$ with domain $P'$, with variable $\eta$ and with negligible function $\pi(\eta)$. It may be that the variables $\xi$ and $\eta$ are independent, so that $\xi$ traverses $N'$ and $\eta$ traverses

[3]) That the variable $\xi$ is "new" means that the number $a$ under consideration is independent of $\xi$.

[4]) Here "constant" means independent of $\xi$.

[5]) Here "variable" means dependent on $\xi$.

$P'$ independently of $\xi$. It is also possible that $\xi$ and $\eta$ are dependent—for instance, that $\xi=\eta$ (then $N'=P'$) or $\xi=\eta^2$ or $\xi=(\varkappa, \lambda)$; $\eta=(\varkappa, \mu)$, where $\varkappa$, $\lambda$, and $\mu$ denote independent variables traversing respectively given sets $K$, $\Lambda$, and $M$. May we neglect all the functions $\nu(\xi)$ belonging to $N$ and at the same time all the functions $\pi(\eta)$ belonging to $P$? Then we also neglect the sum $\nu(\xi)+\pi(\eta)$. The fundamental rule in the neutrix calculus is that we never neglect a constant $\neq 0$. Consequently, the simultaneous neglect of all the functions belonging to $N$ or $P$ is only allowed if these two neutrices satisfy the following condition:

*Whenever it is possible to find in $N$ a negligible function $\nu(\xi)$ and in $P$ a negligible function $\pi(\eta)$ such that the sum $\nu(\xi)+\pi(\eta)$ is constant for each admissible choice of $\xi$ and $\eta$, then this constant is equal to zero.*

Two neutrices with this property are called compatible.

More generally: $s$ neutrices $N_1, \ldots, N_s$ with dependent or independent variables $\xi_1, \ldots, \xi_s$ are called compatible when they satisfy the following condition: *Whenever it is possible to find in $N_\sigma (\sigma=1, \ldots, s)$ a negligible function $\nu_\sigma(\xi_\sigma)$ such that the sum $\sum_{\sigma=1}^{s} \nu_\sigma(\xi_\sigma)$ is constant for each admissible choice of the variables $\xi_1, \ldots, \xi_s$, then this constant is equal to zero.*

The $s$ neutrices with independent variables are always compatible. It is always allowed to neglect simultaneously all the terms which are negligible in compatible neutrices. More precisely:

*If $N_1, \ldots, N_s$ are compatible neutrices with variables $\xi_1, \ldots, \xi_s$ and if two numbers $a$ and $b$, both independent of $\xi_1, \ldots, \xi_s$, have the property that their difference is negligible in $N_1, \ldots, N_s$, then $a$ and $b$ are equal.*

That a number is negligible in $N_1, \ldots, N_s$ means that it can be written as a sum of $s$ terms such that the $\sigma^{\text{th}}$ term $(1 \leqq \sigma \leqq s)$ is negligible in $N_\sigma$.

This fact leads to the following schedule: If it is our task to write a given number $a$ in another form, then we introduce a neutrix $N_1$ with variable $\xi_1$ such that $a$ is independent of $\xi_1$ and we apply the usual rule of calculation with neglect of all the terms which are negligible in $N_1$. If we find in this way that, apart from terms negligible in $N_1$, the number $a$ under consideration is equal to a number $b$ independent of $\xi_1$, then $a=b$ according to the principle formulated above and we have reached our goal. Otherwise we introduce another neutrix $N_2$ compatible with $N_1$ with variable $\xi_2$ such that $a$ is independent of $\xi_2$. Applying again the usual rules of calculation but now neglecting all the terms which are negligible in $N_1$ or $N_2$, we find perhaps that, apart from terms negligible in $N_1$ or $N_2$, the said number $a$ is equal to a number $b$ which is independent of $\xi_1$ and $\xi_2$. In this case $a=b$. Otherwise we introduce a third neutrix $N_3$ compatible with $N_1$ and $N_2$ and so on, until we reach, if possible, our goal.

Let us return to the theory of residues. For the sake of simplicity I restrict myself first to a particular case. Let

$$j = \iint\limits_{\Delta} f(x_1, x_2)\, dx_1\, dx_2$$

be an integral with continuous integrand $f(x_1, x_2)$, extended over a closed bounded integration domain $\Delta$ lying in the real $(x_1, x_2)$-plane. I choose in this domain a finite number, say three of critical points $\alpha_1$, $\alpha_2$, and $\alpha_3$. I consider infinitely many closed subsets $\Delta_1(\xi_1)$ of $\Delta$ which lie in the neighborhood of $\alpha_1$. These subsets $\Delta_1(\xi_1)$ depend on a real variable $\xi_1$ which traverses a given interval $N_1'$. Similarly I introduce infinitely many closed subsets $\Delta_2(\xi_2)$ of $\Delta$ lying in the neighborhood of $\alpha_2$ and infinitely many closed subsets $\Delta_3(\xi_3)$ of $\Delta$ lying in the neighborhood of $\alpha_3$; here $\xi_1$, $\xi_2$, and $\xi_3$ denote independent variables, where $\xi_2$ traverses a given interval $N_2'$ and $\xi_3$ traverses a given interval $N_3'$. If the said neighborhoods of $\alpha_1, \alpha_2$, and $\alpha_3$ are small enough, then $\Delta_1(\xi_1)$, $\Delta_2(\xi_2)$, and $\Delta_3(\xi_3)$ are disjunct. Let $\Delta^*$ denote the set formed by the points of $\Delta$ which belong to none of the sets $\Delta_1(\xi_1)$, $\Delta_2(\xi_2)$, and $\Delta_3(\xi_3)$, so that $\Delta^*$ depends on $\xi_1$, $\xi_2$, and $\xi_3$. We have for the integrand $f(x_1, x_2)$

$$(1) \qquad \jmath = \iint_{\Delta} = \iint_{\Delta_1(\xi_1)} + \iint_{\Delta_2(\xi_2)} + \iint_{\Delta_3(\xi_3)} + \iint_{\Delta^*}.$$

The first term on the right side depends on $\xi_1$, the second on $\xi_2$, the third on $\xi_3$, and the fourth on the three variables $\xi_1$, $\xi_2$, $\xi_3$. It is here that we make the jump by the introduction of neutrices. We assume that it is possible to find a neutrix $N_1$ with domain $N_1'$ and variable $\xi_1$ such that, apart from terms negligible in $N_1$, the first term on the right side of (1) is equal to a number $\gamma_1$ independent of $\xi_1$. Furthermore, we assume that it is possible to find a neutrix $N_2$ with domain $N_2'$ and variable $\xi_2$ such that, apart from terms negligible in $N_2$, the second term on the right side of (1) is equal to a number $\gamma_2$ independent of $\xi_2$. Similarly we assume that is is possible to find a neutrix $N_3$ with domain $N_3'$ and variable $\xi_3$ such that, apart from terms negligible in $N_3$, the third term on the right side of (1) is equal to a number $\gamma_3$ independent of $\xi_3$. Finally we assume that the fourth term on the right side of (1) can be written as a finite sum of terms, negligible in $N_1$, $N_2$, or $N_3$. Then the integral $\jmath$ is equal to $\gamma_1+\gamma_2+\gamma_3$ plus terms which are negligible in $N_1$, $N_2$, or $N_3$. These neutrices are compatible since their variables $\xi_1$, $\xi_2$, and $\xi_3$ are independent. According to the principle of the neutrix calcules we find, therefore,

$$\jmath = \gamma_1+\gamma_2+\gamma_3$$

so that the integral $\jmath$ is equal to the sum of the residues at the critical points $\alpha_1$, $\alpha_2$, $\alpha_3$, if $\gamma_\sigma(\sigma=1, 2 \text{ and } 3)$ is called the residue of the integrand $f(x_1, x_2)$ at the critical point $\alpha_\sigma$. Notice that this residue depends on the choice of the neutrix $N_\sigma$.

As I remarked above, only for the sake of simplicity have I begun with this particular case. It is not necessary that the integration domain be a two-domensional closed bounded set, that the integrand be continuous, and that the number of critical points be equal to 3. As integration domain we can choose any set $\Delta$, and as integrand any real or complex function $f(x)$ defined for each element $x$ of $\Delta$ and integrable over $\Delta$. That

$f(x)$ is integrable over $\Delta$ means that we assign a certain real or complex value to the integral

$$\int_{\Delta} f(x)\,dx,$$

with the understanding that the notion of integral possesses always the two following additive properties:

[1] If $f(x)$ and $g(x)$ are integrable over a set $\Delta$, then their difference $f(x)-g(x)$ is integrable over $\Delta$ and we have

$$\int_{\Delta} (f(x)-g(x))\,dx = \int_{\Delta} f(x)\,dx - \int_{\Delta} g(x)\,dx. \tag{2}$$

[2] If $\Delta$ and $\Delta^*$ are two disjunct sets with union $\Delta+\Delta^*$ and if a function $f(x)$ is integrable over two of the three sets $\Delta$, $\Delta^*$, and $\Delta+\Delta^*$, then $f(x)$ is also integrable over the third of these three sets and we have

$$\int_{\Delta} f(x)\,dx + \int_{\Delta^*} f(x)\,dx = \int_{\Delta+\Delta^*} f(x)\,dx. \tag{3}$$

Property [1] implies: If $f(x)$ and $g(x)$ are integrable over a set $\Delta$, then their sum is integrable over $\Delta$ and we have

$$\int_{\Delta} (f(x)+g(x))\,dx = \int_{\Delta} f(x)\,dx + \int_{\Delta} g(x)\,dx. \tag{4}$$

Indeed, in this case $f(x)-f(x)=0$, $0-g(x)=-g(x)$ and $f(x)+g(x)==f(x)-(-g(x))$ are integrable over $\Delta$ and formula (2), applied with $f(x)$ replaced by $f(x)-g(x)$ yields (4).

Let $s$ be a positive integer. I consider $s$ elements $\alpha_\sigma(\sigma=1, \dots, s)$ of $\Delta$ and moreover $s$ disjunct subsets $A_\sigma(\sigma=1, \dots, s)$ of $\Delta$ such that $\alpha_\sigma$ lies in $A_\sigma$. For convenience I call $A_\sigma$ the neighborhood of $\alpha_\sigma$. Furthermore, we introduce $s$ sets $N_\sigma'(\sigma=1, \dots, s)$ and for each element $\xi_\sigma$ of $N_\sigma'$ we consider a subset $\Delta_\sigma(\xi_\sigma)$ of $A_\sigma$ with the property that $f(x)$ is integrable over $\Delta_\sigma(\xi_\sigma)$. Then the sets $\Delta_\sigma(\xi_\sigma)$ $(\sigma=1, \dots, s)$ are disjunct and $f(x)$ is integrable over the set $\Delta^*=\Delta-\sum_{\sigma=1}^{s}\Delta_\sigma(\xi_\sigma)$, formed by the elements of $\Delta$ which belong to none of the $s$ sets $\Delta_\sigma(\xi_\sigma)$. We have

$$\int_{\Delta} f(x)\,dx = \sum_{\sigma=1}^{s} \int_{\Delta_\sigma(\xi_\sigma)} f(x)\,dx + \int_{\Delta^*} f(x)\,dx. \tag{5}$$

We assume that it is possible to find $s$ neutrices $N_\sigma(\sigma=1, \dots, s)$ with domain $N_\sigma'$ and with variable $\xi_\sigma$ such that

$$\int_{\Delta_\sigma(\xi_\sigma)} f(x)\,dx = \gamma_\sigma+\nu_\sigma(\xi_\sigma) \qquad (\sigma=1, \dots, s),$$

where $\gamma_\sigma$ is independent of $\xi_\sigma$ and where $\nu_\sigma(\xi_\sigma)$ is negligible in $N_\sigma$ and that moreover

$$\int_{\Delta^*} f(x)\,dx$$

can be written as a finite sum of terms each of which is negligible in one of the neutrices $N_1, \dots, N_s$. Then the left side of (5) is equal to $\gamma_1+\dots+\gamma_s$, apart from terms which are negligible in one of the neutrices $N_1, \ldots, N_s$. The variables of these neutrices are independent, so that the neutrices are compatible. Consequently,

$$\int\limits_{\Delta} f(x)\,dx = \gamma_1 + \gamma_2 + \ldots + \gamma_s$$

is the sum of the residues of the integrand at the critical points, if $\gamma_\sigma$ is called the residue at $\alpha_\sigma$.

In the following applications I mention only the results. The integral

$$j_1 = \int\limits_{\alpha_1}^{\alpha_2} x^{\varkappa-1}(1+px^\lambda)^\sigma\,dx,$$

where $\varkappa$ and $\sigma$ are complex numbers, where

$$0 < \alpha_1 < \alpha_2; \quad -\pi < \arg p < \pi; \quad \alpha_3 = |p|^{-1/\lambda}$$

and where $\lambda$ denotes a positive rational number, has 2 or 3 critical points, namely $\alpha_1$ and $\alpha_2$ and perhaps $\alpha_3$; $\alpha_3$ is the third critical point if and only if $\alpha_1 < \alpha_3 < \alpha_2$. If $\alpha_1 < \alpha_3$, then we can choose the corresponding neutrix $N_1$ is such a way that its domain $N_1'$ is the interval $\alpha_1 \leqq \xi_1 < \alpha_3$ and that the residue of the integrand at $\alpha_1$ is equal to

$$-\sum_{h=0}^{\infty} \binom{\sigma}{h} p^h \frac{\alpha_1^{k+\lambda h}}{\varkappa+\lambda h};$$

in this paper $\alpha^\mu/\mu$ denotes $\log \alpha$ in the special case $\mu=0$. If $\alpha_1 > \alpha_3$, then we can choose the corresponding neutrix $N_1$ in such a way that its domain $N_1'$ is the interval $\alpha_1 \leqq \xi_1 \leqq \alpha_2$ and that the residue at $\alpha_1$ is equal to

$$-\sum_{h=0}^{\infty} \binom{\sigma}{h} p^{\sigma-h} \frac{\alpha_1^{\varkappa+(\sigma-h)\lambda}}{\varkappa+(\sigma-h)\lambda}.$$

We obtain a similar result for $\alpha_2$, namely as follows: If $\alpha_2 < \alpha_3$, then we can choose the corresponding neutrix $N_2$ in such a way that its domain is the interval $\alpha_1 \leqq \xi_2 \leqq \alpha_2$ and that the residue at $\alpha_2$ is equal to

$$\sum_{h=0}^{\infty} \binom{\sigma}{h} p^h \frac{\alpha_2^{\varkappa+\lambda h}}{\varkappa+\lambda h}.$$

In the case $\alpha_2 > \alpha_3$ we can choose the corresponding neutrix $N_2$ in such a way that its domain is the interval $\alpha_3 < \xi_2 \leqq \alpha_2$ and that the residue at $\alpha_2$ is equal to

$$\sum_{h=0}^{\infty} \binom{\sigma}{h} p^{\sigma-h} \frac{\alpha_2^{\varkappa+(\sigma-h)\lambda}}{\varkappa+(\sigma-h)\lambda}.$$

If $\alpha_1 < \alpha_3 < \alpha_2$ and if neither $-(\varkappa/\lambda)$ nor $(\varkappa/\lambda)+\sigma$ is an integer $\geqslant 0$, then we can choose the corresponding neutrix $N_3$ in such a way that its domain is the interval $\alpha_1 \leqq \xi_3 \leqq \alpha_2$ and that the residue at $\alpha_3$ is equal to

(6) $$\frac{1}{\lambda}\left[\frac{\varkappa}{\lambda}, \sigma\right] p^{-(\varkappa/\lambda)}, \text{ where } [\mu,\sigma] = \frac{(\mu-1)!\,(-\sigma-\mu-1)!}{(-\sigma-1)!}.$$

The case $\alpha_3 = \alpha_1$, the case $\alpha_3 = \alpha_2$, and the case that at least one of the numbers $-(\varkappa/\lambda)$ and $(\varkappa/\lambda)+\sigma$ is an integer $\geqq 0$, can be treated in a similar way. In this way we come to the conclusion that the integral $j_1$ is always the sum of two or three simple residues.

That $\alpha_1$ and $\alpha_2$ are critical points needs no explanation; in general the endpoints of an integration interval are critical. That in the case $\alpha_1 < \alpha_3 < \alpha_2$ the point $\alpha_3$ is critical needs an explanation, since the integrand is analytic at that point. The explanation is that it is true that at both sides of $\alpha_3$ the integrand possesses expansions according to powers of $x$ but these expansions are not the same, so that at that point the integrand changes its behavior.

Contour integrals can be treated in a similar way. For instance, if the points $\alpha_1$ and $\alpha_2$ lying in the complex $z$-plane are joined by a continuous rectifiable curve which avoids the origin and the points $z$ with $1+pz^\lambda = 0$, if moreover

$$|\alpha_1| < |p|^{-1/(\lambda)} < |\alpha_2|$$

and if finally the said curve contains one and only one point $\alpha_3$ with $|\alpha_3| = |p|^{-(1/\lambda)}$, then the integral

$$j_2 = \int_{\alpha_1}^{\alpha_2} z^{\varkappa-1}(1+pz^\lambda)^\sigma \, dz$$

has the three critical points $\alpha_1$, $\alpha_2$ and $\alpha_3$ and the residues at these points are determined in the same way as in the preceding example.

The integral

$$j_3 = \int_{\alpha_1}^{\alpha_2} x^{\varkappa-1}(1+px^\lambda)^\sigma (1+qx^\mu)^\tau \, dx,$$

where

$$0 < \alpha_1 < \alpha_3 < \alpha_4 < \alpha_2; \quad -\pi < \arg p < \pi; \quad -\pi < \arg q < \pi;$$

$$\alpha_3 = |p|^{-(1/\lambda)}; \quad \alpha_4 = |q|^{-(1/\mu)}$$

and where $\lambda$ and $\mu$ denote positive rational numbers, is more complicated, since it has the four critical points $\alpha_1$, $\alpha_2$, $\alpha_3$, and $\alpha_4$. In order to evaluate the residue at $\alpha_3$, we notice that in the neighborhood of that point the integrand possesses the convergent expansion

$$\sum_{h=0}^{\infty} \binom{\tau}{h} q^{\tau-h} x^{\varkappa-1+(\tau-h)\mu} (1+px^\lambda)^\sigma. \tag{7}$$

Let us assume that neither $(\varkappa+\tau\mu)t$ nor $(\varkappa+\tau\mu+\sigma\lambda)t$ is an integer, where $t$ denotes the smallest positive integer such that $\lambda t$ and $\mu t$ are integers. Then neither $\dfrac{\varkappa+(\tau-h)\mu}{\lambda}$ nor $\dfrac{\varkappa+(\tau-h)\mu}{\lambda}+\sigma$ is an integer. From (6) (applied with $\varkappa$ replaced by $\varkappa+(\tau-h)\mu$) and (7) we can deduce that the residue at $\alpha_3$ is equal to

$$\frac{1}{\lambda}\sum_{h=0}^{\infty} \binom{\tau}{h} \left[\frac{\varkappa+(\tau-h)\mu}{\lambda}, \sigma\right] p^{-\frac{\varkappa+(\tau-h)\mu}{\lambda}} q^{\tau-h}.$$

Also the other residues can be calculated easily.

The following example treats the double integral

$$j_4 = \iint_{\Delta} x^{\varkappa-1}\, y^{\varrho-1}(1+px^{\lambda})^{\sigma}\,(1+qy^{\mu})^{\tau}\, dx\, dy,$$

where

$$-\pi < \arg p < \pi; \quad -\pi < \arg q < \pi;$$

$\lambda$ and $\mu$ are positive rational numbers; $\Delta$ is a triangle lying in the first quadrant $x>0$; $y>0$, with the vertices $\alpha_1, \alpha_2, \alpha_3$. The value of $j_4$ depends on the situation of the point $\alpha_4$ with the coordinates $|p|^{-1/\lambda}$ and $|q|^{-1/\mu}$. Let us assume that this point $\alpha_4$ lies in the interior of the triangle $\Delta$. Let $\alpha_5$ and $\alpha_6$ be the two boundary points of $\Delta$ with abscissa $|p|^{-1/\lambda}$ and let $\alpha_7$ and $\alpha_8$ be the two boundary points of $\Delta$ with ordinate $|q|^{-1/\mu}$. We assume that none of these four points coincides with a vertex of $\Delta$. Then there are eight critical points, namely $\alpha_\sigma(\sigma=1, \ldots, 8)$, so that $j_4$ is equal to the sum of the residues at these eight points. The residue at the vertex $\alpha_1$ depends on the parameters $p, q, \lambda, \mu, \sigma, \tau, \varkappa,$ and $\varrho$ occurring in the integral, on the coordinates of $\alpha_1$ and on the slopes of the two sides of $\Delta$ which come together at $\alpha_1$. The residue at $\alpha_5$ depends on the eight parameters mentioned, on the coordinates of $\alpha_5$, and on the slope of the side on which $\alpha_5$ is lying. The residue at the interior point $\alpha_4$ depends only on the said eight parameters. In particular, if none of the four numbers $-(\varkappa/\lambda)$, $(\varkappa/\lambda)+\sigma$, $-(\varrho/\mu)$ and $(\varrho/\mu)+\tau$ is an integer $\geqslant 0$, then the residue at $\alpha_4$ is equal to

$$(1/\lambda\mu)\,[(\varkappa/\lambda), \sigma]\,[(\varrho/\mu), \tau]\, p^{-(\varkappa/\lambda)}\, q^{-(\varrho/\mu)}.$$

The fact that the definition of neutrices given above involves the notion of equality and not that of asymptotic equality has the consequence that they cannot be used in asymptotics. In view of this circumstance we introduce a new kind of neutrices, asymptotic neutrices, by replacing in the said definition everywhere the concept of equality by that of asymptotic equality. Again $\omega$ denotes an unbounded real or complex variable. Let $N'$ be a set which may depend on $\omega$. Two functions $f(\omega, \xi)$ and $g(\omega, \xi)$, defined for each admissible value of $\omega$ and for each element $\xi$ of $N'$, are called asymptotically equal if for each admissible value of $\omega$, for each element $\xi$ of $N'$, and for each positive fixed $q$ the order relation

$$f(\omega, \xi) - g(\omega, \xi) = O|\omega|^{-q}$$

holds uniformly in $\xi$.

*An asymptotic neutrix $N$ with domain $N'$ and variable $\xi$ is a class of functions $\nu(\omega, \xi)$ defined for each admissible value of $\omega$ and each element $\xi$ of $N'$ with the two following properties:*

[1] *Any two functions $\nu(\omega, \xi)$ and $\nu^*(\omega, \xi)$ belonging to $N$ have the property that $N$ contains at least one function which is asymptotically equal to the difference $\nu(\omega, \xi) - \nu^*(\omega, \xi)$ of these two functions.*

[2] *Each function $\nu(\omega, \xi)$ belonging to $N$ which, apart from a term that is asymptotically zero, is independent of $\xi$ is asymptotically equal to zero.*

Condition [1] implies that $N$ contains also at least one function which is asymptotically equal to the sum $\nu(\omega, \xi)+\nu^*(\omega, \xi)$ of the two said functions.

*The s asymptotic neutrices $N_\sigma(\sigma=1, \ldots, s)$ with variable $\xi_\sigma$ are called compatible if they satisfy the following condition:*

*Whenever it is possible to find in $N_\sigma(\sigma=1, \ldots, s)$ a function $\nu_\sigma(\omega, \xi_\sigma)$ such that the sum $\sum_{\sigma=1}^{s} \nu_\sigma(\omega, \xi_\sigma)$ is for each admissible choice of $\omega, \xi_1, \ldots, \xi_s$ asymptotically equal to a function $\gamma(\omega)$ which is independent of $\xi_1, \ldots, \xi_s$, then $\gamma(\omega)$ is asymptotically equal to zero.*

Asymptotic neutrices with independent variables are always compatible. In asymptotics it is always allowed to neglect terms which are negligible in compatible neutrices. More precisely:

*If $N_1, \ldots, N_s$ are compatible asymptotic neutrices with variables $\xi_1, \ldots, \xi_s$ and if two functions $a(\omega)$ and $b(\omega)$ of $\omega$, both independent of $\xi_1, \ldots, \xi_s$ have the property that their difference is negligible in $N_1, \ldots, N_s$, then $a(\omega)$ and $b(\omega)$ are asymptotically equal.*

As an application we determine for large positive $\omega$ the asymptotic behavior of the integral

$$j = \iint_{\Delta} g(x,y)\, e^{2i\omega xy}\, dx\, dy,$$

where

$$g(x,y) = (x^2+y^2)^{\lambda}\, x^{\sigma} y^{\tau}$$

and where the integration domain $\Delta$ is the trapezoid with the vertices $p_1=(a \cos \varkappa, 0)$; $p_2=(a \cos \varkappa, a \sin \varkappa)$; $p_3=(b \cos \varkappa, 0)$; $p_4=(b \cos \varkappa, b \sin \varkappa)$; here $\varkappa, \lambda, \sigma, \tau, a, b$ denote fixed numbers with $0<\varkappa \leqq (\pi/4)$ and $0<a<b$.

Since $\omega$ is a large positive number, we have here to deal with a rapidly oscillating integrand. The function $2\omega xy$, which is called the phase, is nowhere in the integration domain stationary [6]). The fact that the phase is constant on a part of the boundary—namely, on the base of the trapezoid—makes the problem really difficult. Along the three other sides of the trapezoid the phase is either rapidly increasing or rapidly decreasing. By means of these remarks we can conclude that the four vertices of the trapezoid are the only critical points, so that $j$ is aymptotically equal to the sum of the asymptotic residues at these four points.

To define the asymptotic residue at $p_4$ we consider curves lying in the trapezoid in the neighborhood of $p_4$ on which the phase is constant. The equation of such a curve can be written as

$$2xy = b^2 \sin 2\varkappa - \xi_4,$$

where $\xi_4$ is a small positive number. More precisely: We introduce a fixed [7]) positive number $\delta<1$ and we make the convention that $\xi_4$ is

[6]) Here "stationary" means that both first order partial derivatives with respect to $x$ and $y$ vanish.

[7]) Here "fixed" means independent of $\omega$.

an element of the interval $N_4'$ formed by the positive numbers $\xi_4 < \omega^{-\delta}$. Then $\xi_4$ is small for large $\omega$. It is possible to find a neutrix $N_4$ with domain $N_4'$ and variable $\xi_4$, such that the integral of $g(x, y)\, e^{2i\omega xy}$, extended over the part of the trapezoid formed by the points $(x, y)$ with $2xy \geqq b^2 \sin 2\varkappa - \xi_4$ is asymptotically equal to $\gamma_4 + \nu_4(\xi_4)$, where $\gamma_4$ is independent of $\xi_4$ and where $\gamma_4(\xi_4)$ is negligible in $N_4$. Then $\gamma_4$ is by definition the asymptotic residue at $p_4$.

It is true that this asymptotic residue is defined by means of the neutrix $N_4$, but it can also be defined in another way. Long ago, treating similar but simpler problems (Refs. [5], [6], [7]) I applied smoothing functions which I called neutralizers, since they have the same purpose as neutrices, namely, of neutralization. If $a$ and $b$ are real finite numbers with $a < b$, then I call a function $s(t)$ (this function and the numbers $a$ and $b$ may depend on $\omega$) a smoothing function in $(a, b)$ if it is in the closed interval $(a, b)$ infinitely often differentiable with respect to $t$ with $s^{(h)}(a) = s^{(h)}(b) = 0$ $(h = 1, 2, \ldots)$, if each fixed integer $h \geqslant 0$ satisfies for large $|\omega|$ the order relation

$$s^{(h)}(t) = O\{(b-a)^{-h}\}$$

uniformly in $t (a \leqq t \leqq b)$ and if finally either

$$s(a) = 0; \quad s(b) = 1$$

or

$$s(a) = 1; \quad s(b) = 0.$$

If $s(a) = 0$, then we put $s(t) = 0$ for $t < a$ and $s(t) = 1$ for $t > b$; if $s(a) = 1$, then we put $s(t) = 1$ for $t < a$ and $s(t) = 0$ for $t > b$. Consequently, $s(t)$ is for each real $t$ infinitely often differentiable with respect to $t$.

The classical example is the function $s(t)$ which is for $\alpha \leqslant t \leqslant \beta$ equal to

$$s(t) = c \int_0^{\frac{t-\alpha}{\beta-\alpha}} e^{-(1/v)-(1/1-v)}\, dv,$$

where we choose the constant $c$ such that

$$c \int_0^1 e^{-(1/v)-(1/1-v)}\, dv = 1.$$

Then we have for $\alpha \leqq t \leqq \beta$, uniformly in $t$,

$$s(t) = O(1)$$

and for each fixed integer $h \geqq 1$

$$s^{(h)}(t) = \frac{c}{(\beta-\alpha)^h} \left[ \frac{d^{h-1}}{dv^{h-1}}\, e^{-(1/v)-(1/1-v)} \right]_{v=\frac{t-\alpha}{\beta-\alpha}} = O\left( \frac{1}{(\beta-\alpha)^h} \right)$$

The asymptotic residue $\gamma_4$ at $p_4$ is asymptotically equal to the integral of $g(x, y)\, e^{2i\omega xy}\, s_4(2xy)$, extended over the part of the trapezoid formed by the points $(x, y)$ with

$$2xy \geqq b^2 \sin 2\varkappa - \omega^{-\delta};$$

here $s_4(t)$ is an arbitrary smoothing function in the interval

$$b^2 \sin 2\varkappa - \omega^{-\delta} \leqq t \leqq b^2 \sin 2\varkappa - \tfrac{1}{2}\omega^{-\delta}$$

which vanishes at $t = b^2 \sin 2\varkappa - \omega^{-\delta}$. This means that for the definition of the asymptotic residue at $p_4$ we do not need the theory of neutrices. In this particular case the role of the neutrix $N_4$ can be taken over by the smoothing function $s_4(t)$. But this is not always the case. The smoothing functions (originally called "neutralizers") can take over the role of a particular kind of neutrices, and in the cases where we need neutrices of another kind, for instance in the determination of the asymptotic residues at the vertices $p_1$ and $p_3$, the smoothing functions are too weak. This is the reason why in this domain the asymptotic method exposed in this paper is more powerful than the method based on the smoothing functions.

Below I shall indicate how we determine the asymptotic behavior of the asymptotic residue at $p_4$.

If we replace in this reasoning $b$ by $a$ then we obtain the asymptotic residue at $p_2$ with respect to the triangle with vertices $0, p_1$, and $p_2$. This gives the required asymptotic residue at $p_2$ with respect to the trapezoid, since the sum of these two asymptotic residues at $p_2$ is asymptotically equal to zero.

That the critical points $p_1$ and $p_3$ offer more difficulties than $p_2$ and $p_4$ is a consequence of the fact that the phase is constant on the line segment which joins $p_1$ and $p_3$. There the method applied above cannot be used since neither in the triangle $\Delta^*$ nor in the trapezoid $\Delta$ is it possible to find a branch of an hyperbola $2xy = \text{constant}$ which lies wholly in the neighborhood of $p_1$ or $p_3$.

To define the asymptotic residue at $p_1$ we consider the part of the trapezoid formed by the points $(x, y)$ with

$$x \leqq \alpha \cos \varkappa + \xi_1'; \quad 2xy \leqq \xi_1''; \tag{8}$$

here $\xi_1'$ and $\xi_1''$ denote numbers with

$$\tfrac{1}{2}\omega^{-\varepsilon'} \leqq \xi_1' \leqq \omega^{-\varepsilon'}; \quad \tfrac{1}{2}\omega^{-\varepsilon''} \leqq \xi_1'' \leqq \omega^{-\varepsilon''}, \tag{9}$$

where $\varepsilon'$ and $\varepsilon''$ denote fixed positive numbers with $\varepsilon' + \varepsilon'' < 1$.

We introduce a neutrix $N_1$ with variable $\xi_1 = (\xi_1', \xi_1'')$, where $\xi_1'$ and $\xi_1''$ denote independent variables which traverse the intervals mentioned in(9). Consequently, the doman $N_1'$ of $N_1$ is the rectangle defined in (9). We can choose this neutrix $N_1$ in such a way that the integral of $g(x, y)\, e^{2i\omega xy}$, extended over the part of the trapezoid formed by the points $(x, y)$ with (8), is asymptotically equal to $\gamma_1 + \nu_1(\xi_1)$, where $\gamma_1$ is independent of $\xi_1$ and where $\nu_1(\xi_1)$ is negligible in $N_1$. Consequently, $\gamma_1$ is by definition the asymptotic residue at $p_1$. Notice that we may use the two neutrices simultaneously; indeed they are compatible, since the variables $\xi_4$ and $\xi_1$ are independent.

The asymptotic residue at $p_1$ with respect to the triangle $\Delta^*$ is asymptotically equal to $-\gamma_1$ and if we replace everywhere $a$ by $b$, then $-\gamma_1$ becomes the asymptotic residue at $p_3$. In this way we have defined the asymptotic residue at each of the vertices of the trapezoid.

By means of general rules belonging to the neutrix calculus we can determine the asymptotic behavior of these asymptotic residues. One of these rules is the principle of complex transformation which I will apply to determine the asymptotic behavior of the asymptotic residue at $p_4$. First I apply the transformation

$$u = b \cos \varkappa - x; \quad v = x \sin \varkappa - y \cos \varkappa;$$

hence

$$x = b \cos \varkappa - u; \quad y = b \sin \varkappa - u \tan \varkappa - \frac{v}{\cos \varkappa} \tag{10}$$

which carries the angle of $\Delta$ with vertex $p_4$ into the first quadrant $u \geqq 0$; $v \geqq 0$ of the $uv$-plane. Then we have

$$xy = \tfrac{1}{2} b^2 \sin 2\varkappa - 2ub \sin \varkappa - vb + u^2 \tan \varkappa + \frac{uv}{\cos \varkappa}. \tag{11}$$

Here we have applied a real transformation, and it is easy to see that the asymptotic residue $\gamma_4$ at $p_4$ is asymptotically equal to the asymptotic residue at the origin with respect to the first quadrant $u \geqq 0, v \geqq 0$ of the function

$$-\frac{1}{\cos \varkappa}\, g(x, y)\, e^{2i\omega xy},$$

where $x$ and $y$ denote the functions of $u$ and $v$ defined in (10). Not obvious is the fact that now we can apply with success a complex transformation. The asymptotic behavior of the asymptotic residue at $p_4$ is for given neutrix $N_1$ uniquely defined by the behavior of the integration domain in the neighborhood of that vertex. We can therefore restrict ourselves to small values of $u$ and $v$ so that, apart from the constant term $\tfrac{1}{2} b^2 \sin 2\varkappa$, the preponderant terms occurring on the right side of (11) are the two linear terms $-2ub \sin \varkappa$ and $-vb$. Now we apply a linear complex transformation which expresses $u$ and $v$ by means of two new variables $t$ and $w$ such that these two linear terms are positive for positive $t$ and $w$. I choose

$$u = -it\omega^{-1} \text{ and } v = -iw\omega^{-1},$$

so that in all I apply the complex transformation

$$x = b \cos \varkappa + it\omega^{-1}; \quad y = b \sin \varkappa + it\omega^{-1} \tan \varkappa + iw\omega^{-1} (\cos \varkappa)^{-1}$$

By means of this formal transformation we obtain an expression of the form

$$-(\cos \varkappa)^{-1}\, e^{i\omega b^2 \sin 2\varkappa}\, \omega^{-2} \iint \psi(t, w)\, e^{-4tb \sin \varkappa - 2wb}\, dt\, dw,$$

where

$$\psi(t,w) = g\left(b \cos \varkappa + \frac{it}{\omega}, b \sin \varkappa + \frac{it \tan \varkappa}{\omega} + \frac{iw}{\omega \cos \varkappa}\right) e^{-2it^2\omega^{-1}\tan\varkappa - 2itw\omega^{-1}(\cos\varkappa)^{-1}}.$$

This function possesses for given $t$ and $w$ and for large $\omega$ an asymptotic expansion which is a power series in $\omega^{-1}$. The coefficient of $\omega^{-h}$ in this expansion is a polynomial $\varphi_h(t,w)$ in $t$ and $w$ of degree $\leqq 2h$. It is easy to calculate successively these polynomials. Now the remarkable fact: The principle of complex transformation shows that the asymptotic residue at $p_4$ has the asymptotic expansion

$$-(\cos \varkappa)^{-1} e^{i\omega b^2 \sin 2\varkappa} \sum_{h=0}^{\infty}{}' \omega^{-h-2} \int_0^\infty \int_0^\infty e^{-4tb\sin\varkappa - 2wb} \varphi_h(t,w)\, dt\, dw.$$

We find, therefore, apart from the factor $e^{i\omega b^2 \sin 2\varkappa}$, for the asymptotic residue at $p_4$, an asymptotic expansion according to $\omega^{-2}, \omega^{-3}, \ldots$ with elementary fixed coefficients.

The neutrix calculus is young—only two years old—and the residue calculus is younger—only three months old. "Neutricians" are needed to employ and develop them.

## REFERENCES


1. Corput, J. G. van der, Neutrics, S.I.A.M.J., 7, 3, 253–279 (1959).
2. ———, Introduction to the Neutrix Calculus, Reports 128, 129 and 130, U.S. Army Mathematics Research Center, Madison, Wis., 1960 (to be published in Le Journal d'Analyse Mathématique, 7, 1960).
3. ———, Neutrix Calculus I, II, III, Reports 142 and 144, Report 143 (unpublished), U.S. Army Mathematics Research Center, Madison, Wis., 1960 [Reports 142 and 143 to be published in Proceedings of the Royal Academy of the Netherlands (Proc. Kon. Ned. Akad. v. Wetensch.) 1960].
4. ———, Distributions With Compatible Neutrics, Report (unpublished), U.S. Army Mathematics Research Center, Madison, Wis. (to be published in Le Journal d'Analyse Mathématique, 8, 1960).
5. ———, Zur Methode der Stationären Phase, Composito Mathematica, 1, 15–38 (1934); 3, 328–372 (1936).
6. ———, Sur la Méthode des Points Decisifs, Proc. Kon. Ned. Akad. v. Wetensch., 42, 468–475 (1939); Indagationes Math., 1, 135–142 (1939).
7. ———, On the Method of Critical Points, Proc. Kon. Ned. Akad. v. Wetensch., 51, 650–658 (1948).